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The discovery of localized vibrational modes of a solid wedge, or "wedge acoustic waves," is an important recent advance in the physics and acoustics of solids. Propagating along the edges of wedges, which are the simplest mathematical models of abrupt changes in the orientation of surfaces, these modes contribute to the low-temperature specific heat of the solid (as surface phonons do). This component of the specific heat becomes increasingly important with increasing ratio of the total length of the lines along which the change in orientation occurs to the surface area or volume of the sample. On the other hand, the high concentration of wedge-wave energy near an edge, the absence of dispersion, the absence of diffraction loss, and the relatively low phase velocity make these waves extremely attractive for applications in acoustoelectronic data processing, particularly in nonlinear devices.

Of primary interest here are the so-called antisymmetric wedge waves, in which the vibrations are similar to bending vibrations of plates. In the general case of arbitrary wedge angles, these waves can be studied only by numerical methods (Refs. 11, 12). For acute-angle wedges, however, for which the low-frequency bending modes can be described approximately by the equation of a thin plate of variable thickness, there is an exact (although complicated) analytical solution of the corresponding "inexact" boundary-value problem. This approximate solution expresses the displacements in the wedge waves in terms of special functions. There is also a simpler model theory, which is based on solution of a two-dimensional Helmholtz equation with a wave number which varies in the plane of the plate. For this theory, however, there is no logical basis for the use of a Laplacian in cylindrical coordinates, and there are special functions in the solution, as in Ref. 5.

Our purpose in the present study was to develop a geometrical-acoustics, or ray-tracing, approach to the description of antisymmetric localized vibrational modes of a solid wedge. As we will show below, the corresponding ray-tracing theory, although approximate, is distinguished by its logical consistency. The simplicity of solution and the physical transparency of this theory mean that it can be used to analyze more complex situations corresponding to real solid structures.

We start from a two-dimensional equation for the bending modes of a thin plate which is nonuniform along the x direction. This plate is our acute-angle wedge (Fig. 1), and we are considering the situation not too far from an edge:

\[
\frac{\partial^2}{\partial x^2} [D(x) \left( \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^2 u}{\partial y^2} \right)] + 2(1 - \beta) \frac{\partial^2}{\partial x \partial y} \left( D(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial y^2} [D(x) \left( \frac{\partial^2 u}{\partial y^2} + \gamma \frac{\partial^2 u}{\partial x^2} \right)] - w^2 \rho \phi(x, y) = 0. \tag{1}
\]

Here \( w \) is the normal displacement of the median plane of the plate, \( D(x) = \rho h^3(x) / 12(1 - \sigma^2) \) is the local bending stiffness, \( h(x) \) is the local thickness of the plate, \( E \) and \( c_0 \) are the Young's modulus and Poisson's ratio of the plate material, \( \rho \) is the density of the plate, and \( w \) is the angular frequency. Assuming that the angle \( \theta \) is sufficiently acute, we can simplify the expression for the local thickness of the wedge, \( h(x) = 2x \tan \left( \theta(\pi/2) \right) = 8x \), and we can introduce some notation which will prove convenient below: \( D(x) = \rho h^3(x) + \sigma \rho^2 x^2 \), where \( \sigma = E/12(1 - \sigma^2) = c_0^2 / 12, \sigma = 2c_0 \), (1 - \( \rho(c_0) \) is the velocity of longitudinal waves in the thin plate, and \( c_0 \) and \( c_1 \) are the velocities of longitudinal and shear acoustic waves in an unbounded medium.

We seek a solution of Eq. (1) in the usual representation for geometrical acoustics (or optics) (Refs. 12, 15, 16):

\[
w = A(x) \exp \{ikS(x, y)\}. \tag{2}
\]

where \( A(x) \) and \( S(x, y) = S(x) + (S/k_p) y \) are the slowly varying amplitude and eikonal of the wave.
and $\xi$ is the (conserved) projection of the wave vector of this wave onto the y axis.

Substituting (2) into (1) and equating the real part to zero, we find, in the leading approximation of the eikonal equation,

$$|\psi(z, y)|^2 = \frac{\omega^2}{\varepsilon^2\omega^2} - \frac{k_p^2(z)}{k_p^2} = n^2(z),$$

where $k_r(z) = \sqrt{\omega^2\varepsilon^2 + \omega^2 k_p^2(z)}$ is the local wave number of the bending wave in the plate, $k_p^2 = \omega^2 c_p^2$, and $n_p(x)$ is the corresponding refractive index.

A solution of Eq. (3) corresponding to wave propagation in the positive direction along the x axis is

$$\psi(z) = (1 + i) \left[ |k_r(x)| - \frac{\beta}{2} \right] e^{ix}. \tag{4}$$

Equating the imaginary part to zero, we find the so-called transport equation for the case at hand:

$$\frac{\partial}{\partial z} \left[ \ln|A(z)| \right] + \frac{\partial}{\partial x} \left[ 1 + \frac{1}{2} \left( \frac{\beta}{|k_r|} \right)^2 \right] + \frac{3}{2} \frac{\partial^2}{\partial x^2} = 0. \tag{5}$$

Using (4) to solve Eq. (5), we find

$$A(z) = \frac{G}{\sqrt{|k_r(z) - \frac{\beta}{2}|}}. \tag{6}$$

where $G$ is an arbitrary constant. For normal incidence ($\beta = 0$) we find from (6) the result $A(x) = 1$.

It is simple to verify that expression (6) satisfies the law of energy conservation of the bending waves which propagate through various cross sections of the plate of variable thickness. According to this law, the x component of the energy flux vector of the bending wave, $P_x$, must be a constant for the various cross sections h(x). Since we have $P_x = \psi^* \psi G^2$, where $\psi G^2$ is the x component of the group velocity of the wave and $\psi$ is the phase of the wave, $P_x = \psi^* \psi G^2 / 2$ is the total energy density of the waves in the plate at the given point x, and since we have $\psi G^2 = \delta w / 8 k_p^2 \Psi(z, y)$, in the case where $k_p = \sqrt{k_r^2(x) - \xi^2}$, and $h(x) = \delta x$, we find the relation

$$P_x = \frac{\delta w}{8 \kappa^2} \left[ G^2 \frac{\partial^2}{\partial x^2} (|k_r(z)| - \frac{\beta}{2}) \right] + 2 k_r(z)^2 (\alpha, \sigma, \xi) \left[ |k_r(z)| - \frac{\beta}{2} \right] \kappa_r(z) = G^2 \frac{\partial^2}{\partial z^2} (|k_r(z)| - \frac{\beta}{2}),$$

where expresses energy conservation.

An estimate of the terms which have been discarded shows that the geometrical-acoustics solution in (2), (4), and (6) is valid under the condition

$$k_r(z) \gg 1. \tag{7}$$

In other words, this solution breaks down in the immediate vicinity to the edge of the wedge (at small values of $k_p x$) and/or at large wedge angle $\theta$. Furthermore, the customary limitation of ray-tracing theory follows directly from (6): The observation point must be far from the ray turning points [determined from the condition that the term in square brackets in (6) vanishes] or far from caustic regions. In the case at hand, there are caustics for rays which propagate away from the edge in the positive x direction, i.e., out of a region of lower local bending-wave velocity into a region with a higher one. We should of course not forget that far from the wedge, specifically, under the condition $h(x) = 6T \gg 2T/k_p$, the solution in the form in (2), (4), and (6) again becomes meaningless, since in this case the wedge becomes arbitrarily thick. and the equation for the bending modes of a thin plate, i.e., Eq. (1), no longer applies. We will accordingly be interested below in only one of these solutions of this problem which are concentrated in the region $x \approx 2T/k_p$. In principal, this new approach might be extended to the region $x \approx 2T/k_p$ through the use of, for example, Mindlin's refined theory of plates.\(^{17}\) In this case, however, the analysis would become quite complicated because it would become necessary to justify the neglect of the energy conversion from the lowest bending mode of the plate into higher-index modes (by analogy with the adiabatic approximation in oceanic acoustics\(^{16,11}\)).

To derive a geometrical-acoustics theory of antisymmetric vibrations of a solid wedge which are localized near an edge, we use ray-tracing solution (2), (4), and (6), and we assume that the quantity $\xi$ in this solution is the wave number which we are seeking for the wedge mode. Using the known procedure for the geometrical-acoustics calculation of waveguide modes in a near-surface acoustic channel (Refs. 15 and 16, for example), noting that the phase shift of a bending wave upon its reflection from the edge is $\pi/2$ (Refs. 17 and 18), and noting that the phase shift upon reflection from a simple caustic is $\pi/2$ (Refs. 15, 18, and 19), we can immediately write an implicit dispersion relation for antisymmetric wave modes as a Bohr-Sommerfeld quantization condition:

$$\int \left[ |k_r(z)| - \frac{\beta}{2} \right] \frac{\partial^2}{\partial z^2} = m_l. \tag{8}$$

here $x_1 = 2 \sqrt{3} k_p / \theta_2$ is the coordinate of the turning point for the ray, and $n = 1, 2, 3, \ldots$. The integral on the left side of (8) is tabulated; an evaluation of it leads to a simple expression for the wave numbers of the wedge waves,

$$\xi = \frac{\sqrt{3} k_p}{2}, \tag{9}$$

and a simple expression for the phase velocities of these waves,

$$c = \frac{\omega}{k_p} = \frac{\omega}{\sqrt{3} k_p}. \tag{9}$$

Expression (9) for the velocities of wedge waves, which is obviously independent of the frequency and depends solely on the wedge angle $\theta$, is extremely close to the 'exact' solution of Ref. 5, tending toward it asymptotically with increasing mode index $n$. To see it, we replace the quantity $n$ by $n - 1$ in Eq. (4.6) of Ref. 5, and we replace $\theta$ by $\pi/2$, to change the notation to that of the present paper. We then find the expression $c = c_p^0 [4(1 - \sigma - 4 + 6(1 - \sigma)^2)]^{1/2} / 2\sqrt{3}$, for the velocity, which proves our assertion. Even for the lowest mode ($n = 1$), the discrepancy is quite insignificant, on average (ranging from 22% at $\sigma = 0$ to -13% at $\sigma = 1$), and it disappears completely for $\sigma = 1/3$ [the geometrical-acoustics solution in (9) does not depend on Poisson's ratio $c$ at all]. If we then use $c_p^0 / \sqrt{3} = c_p$ for $c = 1/3$, where $c_p$ is the velocity of a Rayleigh wave, we find that the geometrical-acoustics

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expression for the velocity in (9) becomes the same, at small angles $\theta$, as the expression $c = c_0 \sin(\pi n)$, which has been derived previously through an approximation of curves calculated numerically (Fig. 2).

By way of criticism of these results, we should state that the use of expression (4) in the integrand in (8) near $x = 0$ obviously contradicts condition (7), which states that geometrical-acoustics solution (2), (4), (6) becomes meaningless as $x \to 0$. For this reason, the convergence of integral (8) and the good agreement of the results found from it and the known solutions should be thought of to a large extent as simply "good luck." Another shortcoming of this solution, which is equally as well as shortcoming of Ref. 5, is that far from the edge the wedge can be longer be thought of as a thin plate. If we assume that the depth to which the field of localized modes penetrates into the interior of the wedge is determined by the coordinates of the corresponding ray turning points, $x_t = 2\sqrt{3}\kappa p/\pi$ $= (2/\sqrt{3})n^2 \theta/k_0$, we easily see that the aforementioned inequality $x < 2\sqrt{3}\kappa p/\pi$ places an upper limit on the mode indices: $n^2 < \pi/\theta^2$. Consequently, expression (9) and the corresponding expression of Ref. 5 become inaccurate at large values of $n$. On the other hand, a lower limit on the mode index, $n^2 > 1$, follows from condition (7) for (9). Expression (9) is thus valid for modes with indices $n$ beginning at $n = 2$ and 3 and ending at $n \leq \pi/\theta$.

Through a superposition of quasiplane waves (2), with allowance for (4), (6), and the phase shifts upon reflection, we can easily construct, by a known procedure, the amplitude distribution of the field of antisymmetric vibrations in the transverse direction, $W(x)$. As a result, the overall expression for the displacement amplitudes of wedge waves, $w(x, y) = W(x) \exp(i\gamma y)$, can be written at $x < x_t$ as follows:

$$w(x, y) = \frac{W}{i\kappa \sqrt{\gamma}} L \cdot x \cdot \cos\left[\frac{\sqrt{3}}{2}k_0\left(\frac{x}{\theta} - \frac{y}{\sqrt{3}}\right)^{1/2}\right]$$

where $L = \left[-\frac{k_0}{\sqrt{3}}(x^2 + y^2)^{1/2}\right]$ and $W = \left(1 - \frac{k_0}{\sqrt{3}}x^2\right)^{-1}$.

For values of $x$ which lie beyond the turning points $x_t$ (beyond the caustic), the field distribution $W(x)$ changes, falling off exponentially with increasing $x$. We will not go into a discussion of the related points; we will instead concern ourselves with the more interesting region $x < x_t$, characterized by relation (10).

Calculations of the behavior $W(x)$ from (10) for the first three modes in accordance with the conditions $1 < n^2 < \pi/\theta^2$ established above reveal a fairly good qualitative agreement with the corresponding behavior found in Ref. 5 for the $n = 2$ and 3 modes, a large fraction of whose energy is concentrated in the region of large values of $k_0 x/\theta$. This circumstance reflects the fact (which we have already mentioned) that the velocity $c$ in (9) tends quite rapidly toward the "exact" value with increasing mode index $n$. In contrast with the velocity, which is in excellent agreement with its "exact" value, the transverse field distribution agrees only qualitatively with the results of Ref. 5. Obvious discrepancies are found, in complete accordance with (6) and (7), for values of $k_0 x/\theta$ which correspond to the region near the edge, where the field blows up, and to caustic regions for the corresponding modes, where there is also an unbounded increase in amplitude. Note, however, that, even in the "main" region of validity of the ray-tracing approach which we have taken here, expression (10) cannot be very good, because, in particular, it lacks a dependence of the field on the coordinate $z$.

Despite these deficiencies of this new approach, it may prove extremely useful for analyzing various cases of the propagation of wedge waves in structures whose geometry differs from that of an ideal wedge. For example, if the wedge is truncated, and forms a regular trapezoid (Fig. 1), the phase velocity of the waves can also be determined from (8), but in that expression we would have to carry out the integration over $x$ from the truncation height $x_t$, rather than from zero. The integration leads to the following equation for the velocity $c$:

$$c = \frac{c_0}{\pi} \left(1 - \frac{\sqrt{3}k_0x_t}{c_0} \right)^{1/2}$$

where $c_{0} = c_{0}\theta/\sqrt{3}$ is the velocity in an ideal wedge [see (9)]. Figure 3 shows curves of $c/c_0$ versus $\sqrt{3}\kappa p/\theta$ for the first three modes according to a numerical solution of Eq. (11) by the dichotomizing search method. The truncation is seen to lead to dispersion of the wedge waves. Shown for comparison in this figure are curves (the dashed curves) which follow from a modification of the "exact" solution for the case of an ideal wedge, which can be...
found relatively easily in this case. It is not difficult to see that while the approximate behavior that follows from (11) does not agree at all well the "exact" behavior for the $n = 1$ mode, the agreement is completely satisfactory (as expected) for the second and third modes at large values of $\sqrt{3k_p} l/\theta$.

The geometrical-acoustics approach also makes it an extremely simple matter to analyze a very important problem which has not been discussed previously: the propagation of localized bending waves along the edge of a wedge which is curved in its own plane (the solution found as a result should of course not be regarded as rigorous proof of the existence of such waves). Let us assume that the radius of curvature is positive (a convex edge) and has a value $r_e$. Transforming to cylindrical coordinates, in which the edge of the wedge is described by the equation $r = r_e$, we can then rewrite our governing relation (8) as

$$r^2 \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \right] \phi(r) = - \frac{1}{r} \frac{d}{dr} \left[ m \phi(r) \right].$$

Assuming the radius of curvature $r_e$ to be large ($r_e \gg r_0$, $r_1 - r_0$), and introducing $\ell = r_1 - r$, we can transform from expression (12) to the approximation

$$i \epsilon \left( 1 + \frac{1}{\ell} \right) \left[ \frac{1}{r} \frac{d}{dr} \right] \phi(r) = - \frac{1}{r} \frac{d}{dr} \left[ m \phi(r) \right] = m \phi(r),$$

where the coordinate of the turning point is $\ell_t = \ell_{t_1} (1 - 2\ell_1/r_0)$, where $\ell_{t_1} = 2\sqrt{3k_p} \theta^2 / \pi^2$ is the turning point in the absence of curvature.

Evaluating the integral in (13), which also reduces to a tabulated integral, we can find an expression for the phase velocity of a wedge wave which is propagating along a convex curved edge:

$$c = \frac{v_{3n}}{\sqrt{3}} \left( 1 + \frac{\sqrt{3} n v^2}{2} \right).$$

In other words, the wedge waves again have dispersion in this case. For a edge with a negative curvature (a concave edge), expression (14) remains valid if we replace $r_e$ by $-r_e$.

There are no fundamental difficulties in generalizing (8) to the case of wedges made of anisotropic crystalline materials (a numerical calculation of the characteristics of wedge waves in crystals, including piezoelectric crystals, has been carried out previously in Refs. 4 and 6). For this purpose, the wave number $k_0(x)$ in (8) should be replaced by $k_0(x, \alpha)$, which reflects the orientation dependence of the velocity of a bending wave in a crystal-like plate (see Ref. 21, for example), and we should express the angle of incidence ($\alpha$ which can be reckoned, in particular, from the edge of the wedge) in terms of $k_0(x, \alpha)$ and $h$.

In summary, this new geometrical-acoustics approach to the description of localized vibrational modes of a solid wedge, while being extremely simple and physically transparent, leads to results which agree fairly well with existing data. This new method is also capable of deriving several new results for cases which are relatively intractable in the existing theories. On the other hand, this new approach should not be overestimated. It should be regarded not as a rigorous theory but as a graphic approximate calculation method, which makes it possible to reexamine the physics of wedge acoustic waves from a new viewpoint.